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Focus wavemode propagation in biaxial anisotropic dielectrics

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Abstract. We investigate in the high-frequency limit the propagation of focus wavemodes in the direction of one of the principal axes of a biaxial anisotropic dielectric and in the neighbourhood of this axis. We find that two different kinds of waves can propagate and we discuss some implications of this result.

1. Introduction

We have just proved [1] that two types of high-frequency focus wavemodes, ordinary and extraordinary, can propagate in uniaxial anisotropic dielectrics. The situation is somewhat more intricate in biaxial anisotropic dielectrics (crystals), so we only consider the propagation of high-frequency focus wavemodes in the direction of one of the principal axes of the biaxial crystal and in the neighbourhood of this axis.

Choosing coordinates along the principal axes of the permittivity tensor, a biaxial anisotropic dielectric is defined by the constitutive relations

$$D_j = \varepsilon_j E_j \qquad j = 1, 2, 3 \qquad \boldsymbol{B} = \mu \boldsymbol{H} \tag{1}$$

in which the permittivity tensor is a function of the frequency (but not the direction of the principal axes) while μ is a constant scalar. The values 1, 2, 3, of the subscript j correspond to the components x, y, z, respectively.

Using (1), the Maxwell equations curl $\boldsymbol{H} = c^{-1}\partial_t \boldsymbol{D}$, curl $\boldsymbol{E} = -c^{-1}\partial_t \boldsymbol{B}$ become

$$\mathrm{curl}\, \boldsymbol{H} = c^{-1}\partial_t(\varepsilon \boldsymbol{E}) \qquad \mathrm{curl}\, \boldsymbol{E} = -\mu c^{-1}\partial_t \boldsymbol{H}. \tag{2}$$

We look for the solutions of equations (2) in the form

$$\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{a}(\boldsymbol{x}, t)\exp[\mathrm{i}\omega\chi(\boldsymbol{x}, t)] \qquad \boldsymbol{H}(\boldsymbol{x}, t) = \boldsymbol{b}(\boldsymbol{x}, t)\exp[\mathrm{i}\omega\chi(\boldsymbol{x}, t)] \tag{3}$$

in which, in agreement with the Courant–Hilbert definition [2] of distortion-free progressing waves, a property satisfied by focus wavemodes, the phase χ is a solution of the characteristic equation of the Maxwell equations [2] that we discuss in the next section.

Letting ∂_β denote differentiation with respect to coordinates of space and time, we get from (3)

$$\partial_\beta \boldsymbol{E} = (\partial_\beta \boldsymbol{a} + \mathrm{i}\omega \boldsymbol{a}\partial_\beta\chi)\exp(\mathrm{i}\omega\chi) \qquad \partial_\beta \boldsymbol{H} = (\partial_\beta \boldsymbol{b} + \mathrm{i}\omega \boldsymbol{b}\partial_b\chi)\exp(\mathrm{i}\omega\chi). \tag{4}$$

We assume ω is large enough so that one may neglect $\partial_\beta \boldsymbol{a}$ and $\partial_\beta \boldsymbol{b}$ with respect to $\mathrm{i}\omega\partial_\beta\chi$, so

$$\partial_\beta \boldsymbol{E} \approx \mathrm{i}\omega \boldsymbol{a}\partial_\beta\chi\exp(\mathrm{i}\omega\chi) \qquad \partial_\beta \boldsymbol{H} \approx \mathrm{i}\omega \boldsymbol{b}\partial_\beta\chi\exp(\mathrm{i}\omega\chi). \tag{4a}$$

Substituting (3) into (2) and using (4a) gives

$$\boldsymbol{b} \wedge \mathrm{grad}\,\chi = (\varepsilon \boldsymbol{a})c^{-1}\partial_t\chi \qquad \boldsymbol{a} \wedge \mathrm{grad}\,\chi = -\mu \boldsymbol{b}c^{-1}\partial_t\chi \tag{5}$$

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and introducing the vectors (defined with respect to the principal axes)

$$w_j = \partial_j \chi / c^{-1} \partial_t \chi \tag{6}$$

equations (5) become

$$\boldsymbol{b} \wedge \boldsymbol{w} = (\varepsilon \boldsymbol{a}) \qquad \boldsymbol{a} \wedge \boldsymbol{w} = -\mu \boldsymbol{b} \tag{7}$$

which is a linear homogeneous systems of six equations for the six unknowns a_j, b_j, with a non-trivial solution if its determinant is zero, a condition that will supply the characteristic equation to be satisfied by the phase χ in (3).

Remark. When χ is a linear function of $\boldsymbol{x}$ and t, $\boldsymbol{w}$ is a constant vector proportional to the wavevector and equations (7) are the four-dimensional Fourier transform of the Maxwell equations.

2. Characteristic equation

Eliminating $\boldsymbol{b}$ from (7) and introducing the refractive indices $n_j^2 = \varepsilon_j \mu$ gives

$$\boldsymbol{w} \wedge \boldsymbol{w} \wedge \boldsymbol{a} + (n^2 \boldsymbol{a}) = 0$$

and transforming the triple vector product, we obtain with $w^2 = w_1^2 + w_2^2 + w_3^2$

$$\boldsymbol{w}(\boldsymbol{w} \cdot \boldsymbol{a}) - \boldsymbol{w}^2 \boldsymbol{a} + (n^2 \boldsymbol{a}) = 0 \tag{8}$$

that is

$$\begin{aligned} &a_1 = w_1(\boldsymbol{w} \cdot \boldsymbol{a})(w^2 - n_1^2)^{-1} \qquad a_2 = w_2(\boldsymbol{w} \cdot \boldsymbol{a})(w^2 - n_2^2)^{-1} \\ &a_3 = w_3(\boldsymbol{w} \cdot \boldsymbol{a})(w^2 - n_3^2)^{-1}. \end{aligned} \tag{8a}$$

Multiplying a_j by w_j and summing gives the condition to be satisfied by $\boldsymbol{w}$ to obtain a non-trivial solution of equation (7)

$$w_1^2(w^2 - n_1^2)^{-1} + w_2^2(w^2 - n_2^2)^{-1} + w_3^2(w^2 - n_3^2)^{-1} = 1 \tag{9}$$

or multiplying (9) by the product of denominators

$$w^2(\boldsymbol{w} \cdot \boldsymbol{n})^2 - (w_1 n_1)^2(n_2^2 + n_3^2) - (w_2 n_2)^2(n_3^2 + n_1^2) - (w_3 n_3)^2(n_1^2 + n_2^2) + (n_1 n_2 n_3)^2 = 0. \tag{9a}$$

When χ is a linear function of $\boldsymbol{x}$ and t, that is, for harmonic plane waves, $\boldsymbol{w}$ is a constant vector and equation (9a), sometimes called the dispersion relation, has been the subject of many important works [3–6] analysing the propagation and refraction of light in crystals. A somewhat more mathematical discussion is given in [2] where it is only assumed that χ is linear in time.

In the general case considered here where no assumption is made on χ, substituting (6) into (9a) transforms this equation into a first-order partial differential equation of fourth degree (that is with terms as $(\partial_\beta \chi)^4$) which is the exact characteristic equation of the Maxwell equations [2]. So, some approximation is needed to make tractable this partial differential equation.

3. Solution of the paraxial characteristic equation

As said in the introduction, we investigate the propagation of focus wavemodes in the direction of a principal axis, here oz, and we are interested in the electromagnetic field in the neighbourhood of this axis. Strictly speaking, we assume $w_1^2 \ll w_3^2$, $w_2^2 \ll w_3^2$ so that we may neglect the terms of order w_3^{-j} for $j \geqslant 2$, that we note $0(w_3^{-j})$. Then, equation (9a) reduces to

$$w_3^2 n_3^2 + w_1^2(n_1^2 + n_3^2) + w_2^2(n_3^2 + n_2^2) - n_3^2(n_1^2 + n_2^2) = 0 + 0(w_3^{-2}). \tag{10}$$

Substituting (6) into (10) gives the paraxial characteristic equation for propagation along oz

$$m_1^{-2}(\partial_x \chi)^2 + m_2^{-2}(\partial_y \chi)^2 + m_3^{-2}(\partial_z \chi)^2 - c^{-2}(\partial_t \chi)^2 = 0 \tag{11}$$

$$m_1^{-2} = n_3^{-2}(n_1^2 + n_3^2)(n_1^2 + n_2^2)^{-1} \qquad m_2^{-2} = n_3^{-2}(n_2^2 + n_3^2)(n_1^2 + n_2^2)^{-1}$$
$$m_3^{-2} = (n_1^2 + n_2^2)^{-1}. \tag{11a}$$

Among the many solutions of equation (11), we consider the two following ones

$$\chi = ct - m_3 z - D^{-1}M^2 \qquad M^2 = m_1^2 x^2 + m_2^2 y^2 \qquad D = a + ct + m_3 z \tag{12}$$

$$\chi^\circ = ct - m_3 z - D^{-1}N^2 \qquad N = m_1 x \cos u + m_2 y \sin u. \tag{12a}$$

In D and N, a, u are arbitrary parameters. To prove that (12) and (12a) are solutions of (11), one has just to note that

$$\partial_x \chi = -2m_1^2 x D^{-1} \qquad \partial_y \chi = -2m_2^2 y D^{-1} \tag{13a}$$
$$\partial_z \chi = -m_3(1 - M^2 D^{-2}) \qquad c^{-1}\partial_t \chi = 1 + M^2 D^{-2} \tag{13b}$$

$$\partial_x \chi^\circ = -2m_1 x \cos u N D^{-1} \qquad \partial_y \chi^\circ = -2m_2 y \sin u N D^{-1} \tag{14a}$$
$$\partial_z \chi^\circ = -m_3(1 - N^2 D^{-2}) \qquad c^{-1}\partial_t \chi^\circ = 1 + N^2 D^{-2} \tag{14b}$$

and to substitute (13) and (14) into (11).

The phase (12) is the generalization to anisotropic dielectrics of that of focus wavemodes [7, 8] propagating in free space while for $u = 0, \pi/2$, the phase (12a) is that of TE and TM electromagnetic components of focus wavemodes [1].

Since condition (9) is satisfied (to the order $0(w_3^{-2})$) with (12) and (12a), we may solve equations (7) in terms of one of the components of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$. So, to get electromagnetic focus wavemodes, one has only to take for this component a scalar focus wavemode solution of the wave equation having (11) as the characteristic equation.

4. High-frequency paraxial focus wavemodes

The paraxial wave equation corresponding to the characteristic equation (11) is [2]

$$(m_1^{-2}\partial_x^2 + m_2^{-2}\partial_y^2 + m_3^{-2}\partial_z^2 - c^{-2}\partial_t^2)\psi = 0 \tag{15}$$

and we prove in the appendix that equation (15) has solutions with phases (12) and (12a)

$$\psi = D^{-1}\exp(\mathrm{i}\omega\chi) \qquad \psi^\circ = D^{-1/2}\exp(\mathrm{i}\omega\chi^\circ). \tag{16}$$

Then, we identify the component E_z of the electric field successively with ψ and ψ° so that, according to (3), the z-component of vector $\boldsymbol{a}$ is

$$a_3 = D^{-1} \qquad a_3^\circ = D^{-1/2} \tag{17}$$

while to get the x and y-components of $\boldsymbol{a}$, one has just to use the first relations (8a)

$$(w_2^2 + w_3^2 - n_1^2) - w_1 w_2 a_2 = w_1 w_3 a_3$$
$$-w_1 w_2 a_1 + (w_1^2 + w_3^2 - n_2^2) = w_2 w_3 a_3. \tag{18}$$

This system is easy to solve and to the order $0(w_3^{-2})$ we get using (17)

$$a_1 = D^{-1} w_1/w_3 \qquad a_2 = D^{-1} w_2/w_3$$
$$a_1^\circ = D^{-1/2} w_1^\circ/w_3^\circ \qquad a_2^\circ = D^{-1/2} w_2^\circ/w_3^\circ. \tag{19}$$

Finally, to get the amplitudes b_j, b_j°, one has just to substitute the previous expressions of a_j and a_j° into the second equation (7). So according to (3), this achieves determination of the high-frequency paraxial focus wavemode propagation along the third principal axis of the anisotropic dielectric. However, to be complete, one has still to give the expressions for w_j and w_j°. From (13a, b), we get at once

$$w_1 = -2m_1^2 x D(D^2 + M^2)^{-1} \qquad w_2 = -2m_2^2 y D(D^2 + M^2)^{-1}$$
$$w_3 = -m_3(D^2 - M^2)(D^2 + M^2)^{-1} \tag{20a}$$

and similarly form (14a, b)

$$w_1 = -2m_1 \cos u N D(D^2 + N^2)^{-1} \qquad w_2 = -2m_2 \sin u N D(D^2 + N^2)^{-1}$$
$$w_3 = -m_3(D^2 - N^2)(D^2 + N^2)^{-1}. \tag{20b}$$

In addition, substituting (20a, b) into the inequalities $w_1^2 \ll w_3^2$, $w_2^2 \ll w_3^2$ delimits the regions of spacetime in which the paraxial approximation holds valid.

Comparison of the functions M^2 and N^2 in the phase χ and χ° suggests naming the corresponding solutions symmetric and asymmetric transverse focus wavemodes. Although only the first are mentioned (only in the case of isotropic media) in the literature, the second present the advantage of less attenuation (in $D^{-1/2}$).

5. Discussion

It was shown in [1] that discarding the high-frequency approximation makes it very difficult to calculate analytical expressions. However, can we dispense with the paraxial approximation which limits calculations to a region not too far from the direction of propagation? To obtain an exact solution of the characteristic equation (9a) would be an important achievement.

The propagation of harmonic plane waves in anisotropic dielectrics generates a great variety of physical processes such as double refraction, conical refraction etc of utmost importance in optics (think, for instance, of the Kerr and Pockels effects) and thoroughly analysed in the past [2–6] through discussions of: polarization, phase velocity, normal surfaces, ray surfaces and so on. Part of this analysis may be generalized to waves with arbitrary phases provided they are linear in time [2], as Gaussian beams. So, it is a bit frustrating to work with focus wavemodes since one has only to be content that they can propagate at least in some direction. Nevertheless, this last possibility could rejuvenate some important problems conventionally tackled with plane waves as, for instance, the behaviour of plasmas excited by electromagnetic waves.

So, in order to know the way that focus wavemodes propagate in different media will not be a futile exercise as soon as one is able to generate [1] physical focus wavemodes, as already realized in acoustics [9], with the potential of being a good approximation of mathematical focus wavemodes in bounded regions of spacetime.

Appendix

To prove that (16) is a solution of equation (15), we first use (13a) then

$$\partial_x \psi = -2\mathrm{i}\omega m_1^2 x D^{-1}\psi \qquad \partial_y \psi = -2\mathrm{i}\omega m_2^2 y D^{-1}\psi \tag{A.1}$$

and a simple calculation gives

$$\partial_x^2 \psi = -(2\mathrm{i}\omega m_1^2 D^{-1} + 4\omega^2 m_1^4 x^2 D^{-2})\psi \qquad \partial_y^2 \psi = -(2\mathrm{i}\omega m_2^2 D^{-1} + 4\omega^2 m_2^4 y^2 D^{-2})\psi. \tag{A.2}$$

Similarly from (13b)

$$\begin{aligned} \partial_z \psi &= -m_3[D^{-1} - \mathrm{i}\omega(1 - M^2 D^{-2})]\psi \\ c^{-1}\partial_t \psi &= [D^{-1} + \mathrm{i}\omega(1 + M^2 D^{-2})]\psi \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} \partial_z^2 \psi &= m_3^2[2D^{-2} + 2\mathrm{i}\omega D^{-1} - 4\mathrm{i}\omega D^{-3} M^2 - \omega^2(1 - M^2 D^2)]\psi \\ c^{-2}\partial_t^2 \psi &= [2D^{-2} - 2\mathrm{i}\omega D^{-1} - 4\mathrm{i}\omega D^{-3} M^2 - \omega^2(1 + M^2 D^{-2})]\psi. \end{aligned} \tag{A.4}$$

Finally from (A.2) and (A.4)

$$(m_3^{-2}\partial_z^2 - c^{-2}\partial_t^2)\psi = [4\mathrm{i}\omega D^{-1} + 4\omega^2 M^2 D^{-2})\psi = -(m_1^{-2}\partial_x^2 + m_2^{-2}\partial_y^2)\psi \tag{A.5}$$

which is exactly equation (15). We proceed similarly for the second solution. From (14a)

$$\partial_x \psi^\circ = -2\mathrm{i}\omega m_1 \cos u N D^{-1}\psi^\circ \qquad \partial_y \psi^\circ = -2\mathrm{i}\omega m_2 \sin u N D^{-1}\psi^\circ. \tag{A.6}$$

In this case also, a simple calculation gives

$$\begin{aligned} \partial_x^2 \psi^\circ &= -m_1^2[2\mathrm{i}\omega D^{-1}\cos^2 u + 4\omega^2 N^2 D^{-2}\cos^2 u]\psi^\circ \\ \partial_y^2 \psi^\circ &= -m_2^2[2\mathrm{i}\omega D^{-1}\sin^2 u + 4\omega^2 N^2 D^{-2}\sin^2 u]\psi. \end{aligned} \tag{A.7}$$

Similarly form (14b)

$$\begin{aligned} \partial_z \psi^\circ &= -m_3[1/2D + \mathrm{i}\omega(1 - N^2 D^{-2})]\psi^\circ \\ c^{-1}\partial_t \psi^\circ &= -[1/2D - \mathrm{i}\omega(1 - N^2 D^{-2})]\psi^\circ \end{aligned} \tag{A.8}$$

$$\begin{aligned} \partial_z^2 \psi^\circ &= m_3^2[3/4D^2 - 3\mathrm{i}\omega D^{-3} N^2 + \mathrm{i}\omega D^{-1} - \omega^2(1 - N^2 D^{-2})]\psi^\circ \\ c^{-2}\partial_t^2 \psi^\circ &= [3/4D^2 - 3\mathrm{i}\omega D^{-3} N^2 - \mathrm{i}\omega D^{-1}\omega^2(1 + N^2 D^{-2})]\psi^\circ \end{aligned} \tag{A.9}$$

and according to (A.7) and (A.9)

$$m_3^{-2}\partial_z^{-2}\psi^\circ - c^{-2}\partial_t^2\psi^\circ = (2\mathrm{i}\omega D^{-1} + 4\omega^2 N^2 D^{-2})\psi^\circ = -(m_1^2\partial_x^2 + m_2^2\partial_y^2)\psi^\circ \tag{A.10}$$

which is equation (15).

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